DEFINITE INTEGRALS

Let ϕ(x) be the primitive or anti derivative of a function f(x) defined on [a, b] i.e., $\frac{d}{dx}(\phi(x)) = f(x)$.

Then the definite integral of f(x) over [a, b] is denoted by $\int_a^b f(x)\,dx$ and is defined as $[\phi(b) - \phi(a)]$ i.e.,

$$\int_a^b \mathbf{f(x)\,dx = \phi(b) - \phi(a)} \qquad \ldots(i)$$

The numbers a and b are called the limits of integration, 'a' is called the **lower limit** and 'b' the **upper limit**. The interval [a, b] is called the **interval of integration**.

If we use the notation $[\phi(x)]_a^b = \phi(b) - \phi(a)$, then from (i),

$$\Rightarrow \int_a^b f(x)\,dx = [\phi(x)]_a^b = [(\phi(x) \text{ at } x = b) - (\phi(x) \text{ at } x = a)]$$

= (value of anti derivative at b, the upper limit) – (value of anti-derivative at a, the lower limit)

Remark - In the above definition it does not matter which anti-derivative is used to evaluate the definite integral because if $\int f(x)\,dx = \phi(x) + C$, then

$$\int_a^b f(x)\,dx = [\phi(x) + C]_a^b = (\phi(b) + C - (\phi(a) + C) = \phi(b) - \phi(a).$$

In other words, **to evaluate the definite integral there is no need to keep the constant of integration.**

EVALUATION OF DEFINITE INTEGRALS

To evaluate the definite integral $\int_a^b f(x)\,dx$ of a continuous functions f(x) defined on [a, b] we use the following algorithm.

ALGORITHM

Step 1: Find the indefinite integral $\int_a^b f(x)\,dx$. Let this be $\phi(x)$. There is no need to keep the constant of integration.

Step 2: Evaluate $\phi(b)$ and $\phi(a)$

Step 3: Calculate $\phi(b) - \phi(a)$. The number obtained in Step III is the value of the definite integral $\int_a^b f(x)\,dx$.

EXAMPLES:

1. Evaluate:

(1) $\int_1^2 x^2dx$ (2) $\int_{-4}^{-1}\frac{1}{x}dx$ (3) $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}dx$ (4) $\int_0^1 \frac{1}{2x-3}dx$

Sol. (1) $\int_1^2 x^2dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3}-\frac{1^3}{3}=\frac{8}{3}-\frac{1}{3}=\frac{7}{3}.$

(2) $\int_{-4}^{-1}\frac{1}{x}dx = [\log|x|]_{-4}^{-1} = [\log|-1|-\log|-4|]$ = log 1 – log 4 = 0 – log 4 = – log 4.

(3) $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}dx = \int_0^1 \frac{\sqrt{1+x}-\sqrt{x}}{(\sqrt{1+x}+\sqrt{x})(\sqrt{1+x}-\sqrt{x})}dx = \int_0^1 \left(\sqrt{1+x}-\sqrt{x}\right)dx$

$= \left[\frac{2}{3}(1+x)^{3/2}-\frac{2}{3}x^{3/2}\right]_0^1 = \left[\frac{2}{3}(1+1)^{3/2}-\frac{2}{3}(1)^{3/2}\right]-\left[\frac{2}{3}(1+0)^{3/2}-\frac{2}{3}(0)^{3/2}\right]$

$= \frac{2}{3}[2^{3/2}-1]-\frac{2}{3}[1-0] = \frac{2}{3}\left[2\sqrt{2}-2\right]$

(4) $\int_0^1 \frac{1}{2x-3}dx = \frac{1}{4}[\log(2x-3)]_0^1 = \frac{1}{2}[\log \left||-1|-\log|-3|\right|]$

$= \frac{1}{2}[\log 1 - \log 3] = \frac{1}{2}[0-\log 3] = -\frac{1}{2}\log 3.$

2. Evaluate:

(i) $\int_0^{\pi/4} \tan^2 x\,dx$ (ii) $\int_0^{\pi/2} \sin^2 x\,dx$ (iii) $\int_0^{\pi/4} \sin 3x \sin 2x\,dx$

Sol. (i) $\int_0^{\pi/4} \tan^2 x\,dx = \int_0^{\pi/4} (\sec^2 x-1)\,dx = [[\tan x - x]]_0^{\pi/4} = \left(\tan\frac{\pi}{4}-\frac{\pi}{4}\right) - (\tan 0 - 0) = \left(1-\frac{\pi}{4}\right)$

(ii) $\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} (1-\cos 2x)\,dx = \frac{1}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}-\frac{\sin\pi}{2}\right)-\left(0-\frac{\sin 0}{2}\right)\right]=\frac{\pi}{4}$

(iii) $\int_0^{\pi/4} \sin 3x \sin 2x\,dx = \frac{1}{2}\int_0^{\pi/4} (2\sin 3x \sin 2x)dx = \frac{1}{2}\int_0^{\pi/4} (\cos x - \cos 5x)dx = \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right]_0^{\pi/4}$

3. Evaluate: $\int_0^{\pi/4} \sqrt{1+\sin 2x}\,dx$

Sol. $\int_0^{\pi/4} \sqrt{1+\sin 2x}\,dx = \int_0^{\pi/4}\sqrt{\sin^2 x+\cos^2 x+2\sin x\cos x}\,dx$

$= \int_0^{\pi/4} (\cos x + \sin x)\,dx = [\sin x - \cos x]_0^{\pi/4}$

$= \left(\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right) - (\sin 0 - \cos 0) = \left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right) - (0-1) = 1$

4. Evaluate $\int_{\pi/4}^{\pi/2} \sqrt{1-\sin 2x}\, dx$

Sol. $\int_{\pi/4}^{\pi/2} \sqrt{1-\sin 2x}\, dx = \int_{\pi/4}^{\pi/2} \sqrt{\cos^2 x + \sin^2 x - 2\sin x \cos x}\, dx$

$\int_{\pi/4}^{\pi/2} \sqrt{(\cos x - \sin x)^2}\, dx = \int_{\pi/4}^{\pi/2} |\cos x - \sin x|\, dx \quad \left[\text{as} \sqrt{x^2} = |x|\right]$

$= \int_{\pi/4}^{\pi/2} -(\cos x - \sin x)\, dx$

$\left[\text{As } \cos x < \sin x \text{ for } \frac{\pi}{4} < x < \frac{\pi}{2} \therefore \cos x - \sin x < 0 \Rightarrow |\cos x - \sin x| = -(\cos x - \sin x)\right]$

$= \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx = [\cos x - \sin x]_{\pi/4}^{\pi/2}$

$\left(-\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right) - \left(-\cos\frac{\pi}{4} - \sin\left(\frac{\pi}{4}\right)\right) = (0-1) - \left(\frac{2}{\sqrt{2}}\right) = \sqrt{2} - 1$

5. If $\int_0^1 (3x^2 + 2x + k)\, dx = 0$, find k.

Sol. $\int_0^1 (3x^2 + 2x + k)\, dx = 0 \Rightarrow [x^3 + x^2 + kx]_0^1 = 0 \Rightarrow (1 + 1 + k) - 0 = 0 \Rightarrow = -2$

6. If $\int_1^a (3x^2 + 2x + 1)\, dx = 11$, find a.

Sol. $\int_1^a (3x^2 + 2x + 1)\, dx = 11 \Rightarrow [x^3 + x^2 + x]_1^a = 11 \Rightarrow (a^3 + a^2 + a) - (1 + 1 + 1) = 11$

7. Evaluate $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x}) dx$

Sol. $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x}) dx = \int_0^{\pi/2} \sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\, dx$

$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx = \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}} dx = \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx$

Put sin x – cos x = t, so that (cos x + sin x) dx = dt.

When x = 0, t = –1, when x = $\frac{\pi}{4}$, t = 1.

$\therefore \int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx = \sqrt{2} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\ [\sin^{-1} t]_{-1}^{1}$

$\sqrt{2}\ [\sin^{-1}(1) - \sin^{-1}(-1)] = \sqrt{2}\ [2 \sin^{-1}(1)] = 2\sqrt{2}\left(\frac{\pi}{2}\right) = \sqrt{2}\,\pi$

8. Evaluate $\int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^3} dx$

Sol. Let I = $\int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^3} dx = \int_0^{\pi/2} \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^3} dx = \int_0^{\pi/2} \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} dx$

Put $\cos\frac{x}{2} + \sin\frac{x}{2}$ = t, so that $\frac{1}{2}\left(-\sin\frac{x}{2}+\cos\frac{x}{2}\right)dx = dt$ or $\left(-\sin\frac{x}{2}-\cos\frac{x}{2}\right) dx = 2dt$.

Also, x = 0 ⇒ t = cos 0 + sin 0 = 1

And $x = \frac{\pi}{2} \Rightarrow t = \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\therefore I = \int_1^{\sqrt{2}} \frac{2dt}{t^2} = 2\int_1^{\sqrt{2}} \frac{1}{t^2} = 2\left[-\frac{1}{t}\right]_1^{\sqrt{2}} = 2\left[-\frac{1}{\sqrt{2}}+1\right] = (2-\sqrt{2})$.

PROPERTIES OF DEFINITE INTEGRALS

Property 1: $\int_a^b f(x)dx = \int_a^b f(t)\,dt$ (change of variable)

Property 2: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (interchanging of limits)

Property 3: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where a < c < b.

Generalization: The above property can be generalized into the following form

$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx \int_{c_1}^{c_2} f(x)dx + \ldots. + \int_{c_4}^{b} f(x)dx$, where $a < c_1 < c_2 < c_3 \ldots\ldots < c_{n-1} < c_n < b$

EXAMPLES

9. $\int_{-1}^{1} f(x)\,dx$, where $f(x) = \begin{cases} 1-2x, & x \le 0 \\ 1+2x, & x \ge 0 \end{cases}$

Sol. We have $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx = \int_{-1}^{0} (1-2x)\,dx + \int_{-1}^{0} (1+2x)\,dx$ [by def. of f(x)]

$= [x - x^2]_{-1}^{0} + [x + x^2]_{-1}^{0} = [0 - (1-1)] + [(1+1) - (0)] = 4$

10. $\int_0^1 |5x-3|\,dx$

Sol. $|5x-3| = \begin{cases} -(5x-3) & \text{when } 5x-3<0 \text{, i.e., } x<\frac{3}{5} \\ 5x-3 & \text{when } 5x-3\ge 0 \text{, i.e., } x\ge\frac{3}{5} \end{cases}$

$$\int_0^1 |5x-3|\,dx = \int_0^{3/5} |5x-3|\,dx + \int_{3/5}^1 |5x-3|\,dx = \int_0^{3/5} -(5x-3)dx + \int_{3/5}^1 (5x-3)dx$$

$$= \left[3x-\frac{5x^2}{2}\right]_0^{3/5} + \left[\frac{5x^2}{2}-3x\right]_{3/5}^1 = \left(\frac{9}{5}-\frac{9}{10}\right)+\left(-\frac{1}{2}+\frac{9}{10}\right)=\frac{13}{10}$$

Property 4: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

EXAMPLE:

11. Evaluate: $\int_0^\pi \frac{x}{1+\sin x}dx$

Sol. Let $I = \int_0^\pi \frac{x}{1+\sin x}dx$ … (i)

Then $I = \int_0^\pi \frac{\pi-x}{1+\sin(\pi-x)}dx$ $\left[\text{as} \int_0^b f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^\pi \frac{\pi-x}{1+\sin x}dx$ … (ii)

Adding (i) and (ii), we get

$2I = \int_0^\pi \frac{x+\pi-x}{1+\sin x}dx = \pi\int_0^\pi \frac{1}{1+\sin x}dx$ … (iii)

$= \pi\int_0^\pi \frac{1-\sin x}{1+\sin x}dx = \pi\int_0^\pi (\sec^2 x - \tan x \sec x)\,dx = \pi\,[\tan x - \sec x]_0^\pi$

$= \pi\,[(\tan\pi - \sec\pi) - (\tan 0 - \sec 0)] = \pi\,[0-(-1)-(0-1)] = 2\pi$.

$\therefore I = \pi$

Property 5: $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)dx$, if f(x) is even function i,e. $f(-x) = f(x)$

$= 0$, if f(x) is an odd function i,e. $f(-x) = -f(x)$

EXAMPLE:

12. $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x \, dx$

Sol. Let $f(x) = x^3 \sin^4 x$.

Then $f(-x) = (-x)^3 \sin^4(-x) = -x^3 (\sin(-x))^4 = -x^3(-\sin x)^4 = -x^3 \sin^4 x = -f(x)$.

So value of integral is zero.

Property 6: $\int_0^{2a} f(x)dx = 2\int_0^a f(x)\,dx$, if $f(2a - x) = f(x)$

$= 0$, if $f(2a - x) = -f(x)$

EXAMPLE:

13. Evaluate $\int_0^{2\pi} \cos^5 x \, dx$.

Sol. Let $f(x) = \cos^5 x$.

Then, $f(2\pi - x) = [\cos(2\pi - x)]^5 = \cos^5 x$

$\therefore \int_0^{2\pi} \cos^5 x \, dx = 2\int_0^{\pi} \cos^5 x \, dx$

Now, $f(\pi - x) = [\cos(\pi - x)]^5 = -\cos^5 x = -f(x)$

$\therefore \int_0^{\pi} \cos^5 x \, dx = 0$.

Hence, $\int_0^{2\pi} \cos^5 x \, dx = 2\int_0^{\pi} \cos^5 x \, dx = 2 \times 0 = 0$.

Property 7: $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

EXAMPLE:

14. Evaluate $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx$.

Sol. Let $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ … (i)

Then, $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$

$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos}+\sqrt{\sin x}}\,dx$ … (ii)

Adding (i) and (ii), we get,

$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx = \int_{\pi/6}^{\pi/3} 1.\,dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$

$\therefore I = \pi/12$.

Property 8: If f(x) is a periodic function with period T, then

(i) $\int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx$ (ii) $\int_a^{a+T} f(x)\, dx$ is independent of a.

EXAMPLE:

15. Prove that $\int_0^{10} (x-[x])\, dx = 5$

Sol. Since x – [x] is a periodic function with period one unit. Therefore,

$$\int_0^{10} (x-[x])\, dx = 10\int_0^{1} (x-[x])dx = 10\left[\int_0^1 xdx - \int_0^1 [x]dx\right]$$

$$= 10\left[\left[\frac{x^2}{2}\right]_0^1 - 0\right] = \frac{10}{2} = 5$$

Property 9: Walli's theorem

$$\int_0^{\pi/2} \sin^n x\, dx \text{ or } \int_0^{\pi/2} \cos^n x\, dx$$

$= \frac{(n-1)(n-3)...2}{n(n-2).........1}$ if n is **odd.**

And $= \frac{(n-1)(n-3).....1}{n(n-2).......2}\cdot\frac{\pi}{2}$ if n is **even.**

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx$$

$= \frac{(m-1)(m-3)....(n-1)(n-3).....}{(m+n)(m+n-2)......}$

(If m, n are **both odd** positive integers **or one odd** positive integer)

$= \frac{(m-1)(m-3)....(n-1)(n-3).....}{(m+n)\,(m+n-2).....}\cdot\frac{\pi}{2}$

(If m, n are **both positive even** integers)

Property 10: Leibnitz's rule for the differentiation under the integral sign:

If the functions ϕ (x) and ψ (x) are defined on [a, b] and differentiable at a point x ∈ (a, b) and f(x, t) is continuous.

(i) $\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)dt\right] = \int_{\varphi(x)}^{\Psi} \frac{\partial}{\partial x} f(x,t)dt + \frac{d\psi(x)}{dx} f(x, \psi(x)) - \frac{d}{dx}(\phi(x) f(x, \phi(x))$

If the functions ϕ (x) and ψ (x) are defined on [a, b] and differentiable at a point x ∈ (a, b), and f (t) is continuous on [ϕ (a), ϕ (b)], then

(ii) $\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)dt\right] = \frac{d}{dx}\{\psi(x)\} f(\psi(x)) - \frac{d}{dx}\{\phi(x)\} f(\phi(x))$

EXAMPLE:

16. Find $\frac{d}{dx}\left[\int_{x^2}^{x^3}\frac{1}{\log t}dt\right]$

Sol. $\frac{d}{dx}\left[\int_{x^2}^{x^3}\frac{1}{\log t}dt\right] = \frac{d}{dx}(x^3).\left(\frac{1}{\log x^3}\right) - \frac{d}{dx}(x^2).\left(\frac{1}{\log x^2}\right)$

$= \frac{3x^2}{3\log x} - \frac{2x}{2\log x} = \frac{1}{\log x}(x^2 - x).$

LIMIT AS A SUM

Algorithm

Step I Obtain the given series.

Step II Express the series in the form $\lim_{n\to\infty}\left[\frac{1}{n}\sum f\left(\frac{r}{n}\right)\right]$

Step III Replace $\sum$ by $\int$, $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx.

Step IV Obtain lower and upper limits by computing $\lim_{n\to\infty}\left(\frac{r}{n}\right)$ for the least and greatest values of r respectively.

Step V Evaluate the integral obtained in previous step. The value so obtained is the required sum of the given series.

EXAMPLE:

17. Show that the sum of the series $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{3n}$ as $n \to \infty$ is log 3.

Sol. Let S = $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+2n}\right]$

$= \lim_{n\to\infty}\left[\sum_{r=0}^{2n}\frac{1}{n+r}\right] = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=0}^{2n}\frac{n}{n+r}\right]$

$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=0}^{2n}\frac{1}{1+(r/n)}\right]$

Now, lower limit = $\lim_{n\to\infty}\frac{r}{n} = \lim_{n\to\infty}\frac{0}{n} = 0$

[$\because$ r = 2n for the first term]

Upper limit = $\lim_{n\to 0}\frac{r}{n} = \lim_{n\to 0}\frac{2n}{n} = 2$

$\therefore$ S = $\lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=0}^{2n}\frac{1}{1+(r/n)}\right] = \int_0^2\frac{1}{1+x}dx = [\log(1+x)]_0^2$

= log 3 – log 1 = log 3.